AIAA JOURNAL
Vol. 35, No. 3, March 1997

Criteria for Occurrence of Flutter Instability Before Buckling in Nonconservative Dissipative Systems

A. N. Kounadis*
National Technical University of Athens, Athens 10682, Greece

The occurrence of flutter instability through a Hopf bifurcation before static buckling in regions of divergence in nonconservative, nonself-adjoint, dissipative systems is thoroughly discussed using a qualitative analysis. This region where both Ziegler's and static criterion may fail to predict the actual critical load is defined via two values (bounds) of the nonconservativeness loading parameter η; the upper bound corresponds to $\eta = 0.5$ (being invariant with respect to all other parameters), whereas the lower bound corresponds to a double critical (divergence) point beyond which there are no adjacent equilibria. The location of the last point (lying always between $\eta = 0$ and 0.5) depends on a stiffness parameter. It is also found that the region of nonexistence of adjacent equilibria becomes maximum (minimum) when the double critical point corresponds to $\eta = 0.5$ ($\eta = 0$). The interaction of vanishing damping with various parameters leads to new phenomena related to point and periodic attractors as well as to a new type of dynamic bifurcation.

I. Introduction

THE loss of elastic stability of nonconservative systems under partial follower loading (nonself-adjoint systems) associated with a nonconservativeness loading parameter η may occur either by divergence (static instability) or by flutter (dynamic instability). There is a large amount of pertinent work in the last 30 years.[1–6] Without restricting the generality of the present study, we may assume that η varies between $\eta = 1$ (conservative load) and 0 (tangential load). Using a classical analysis, one can obtain the critical load in the first case by applying either the static or the kinetic (Ziegler's) criterion (associated with the vanishing of the fundamental circular frequency), whereas one can obtain the critical load in the second case only by employing the kinetic criterion.

With the aid of some basic concepts of the theory of dynamical systems and using a qualitative analysis, the effect of interaction of small (and vanishing) damping with various geometric parameters on the precritical, critical, and postcritical response is thoroughly discussed. Attention is focused on the establishment of conditions for the existence of local dynamic bifurcations in regions of existence of adjacent equilibria. In this case, if dynamic bifurcations may occur prior to static bifurcations, then the static criterion and the dynamic (Ziegler's) criterion fail to predict the actual critical load. Some cases where this phenomenon may occur have been reported recently by Kounadis.[7] The present study extending the last work has the following objectives: 1) to find out which parameters may affect the aforementioned phenomenon, 2) to establish the range of values of these parameters for which the static and dynamic (Ziegler's) criterion fail to predict the actual critical load in regions of divergence, 3) to find out whether there are parameters whose variation may render maximum and minimum the region of validity and failure of the aforementioned criteria, and 4) to reveal all possible types of dynamic bifurcations in the region of divergence.

II. Mathematical Analysis

Consider a nonlinear, geometrically perfect, dissipative system under a partial follower load λ associated with a nonconservativeness parameter η, described by a finite set of variables y_i ($i = 1, \ldots, n$), where $y_i = y_i(t)$ represents the state of the system at time t. The response of such an autonomous nonpotential system evolves according to the following matrix-vector form equation:

$$\frac{\mathrm{d}y}{\mathrm{d}t} \overset{\text{def}}{=} \dot{y} = \boldsymbol{Y}(\boldsymbol{y}, \boldsymbol{\mu})\boldsymbol{y} \in R^n, \qquad \mu \in R^m \qquad (n = \text{even}) \quad (1)$$

where $\boldsymbol{y} = (y_1, \ldots, y_n)^T$ is the state vector in the Euclidean space R^n with T denoting transpose; $\boldsymbol{Y}$ is a nonlinear vector function, whereas $\mu = (\mu_1, \ldots, \mu_m)^T$ is the (control) parameter vector with m independent parameters such as λ, η, k_i (stiffnesses with $i \leq n/2$), etc. The term $\boldsymbol{Y}(\boldsymbol{y}, \mu)$ is assumed to be analytic in the state vector $\boldsymbol{y}$ and in the control parameter vector μ, at least in the domain of interest. The $(n + m)$ dimensional state-parameter space, R^{n+m}, spanned by y_i and μ_j, is assumed to be Euclidean space. Hence, the state-space R^n and the parameter space R^m are subspaces of the state-parameter space R^{n+m}. There is one-to-one correspondence between the set of variables $(\boldsymbol{y}, \mu)$ and the points of R^{n+m}. Static and dynamic bifurcations as well as instability of equilibria and limit cycles occur at certain values of these parameters that are assumed to vary smoothly. By dynamic bifurcation we define a sudden qualitative change of the system dynamic response occurring at a certain value of a smoothly varying control parameter. From the viewpoint of topology the set of bifurcations of Eq. (1) corresponds to those values of μ for which $\boldsymbol{Y}(\boldsymbol{y}, \mu)$ becomes a structurally unstable dynamical system,[8] that is, when the phase portrait is changed to a topologically nonequivalent portrait by a smooth change of the control parameter. It is also assumed that the bifurcations (static or dynamic) lie on a trivial fundamental (precritical) equilibrium path.

Attention is focused on steady-state solutions of Eq. (1) (i.e., equilibria and periodic motions) associated with point and periodic attractors, namely, on the long-term response after the transients disappear (decaying to zero at $t \to \infty$). The cases of quasiperiodic (torus) and strange (chaotic) attractors are excluded for this type of loading. The system behavior under the influence of various parameters can jump from one solution to another at bifurcation (critical) points occurring at certain values of the smoothly varying control parameters; i.e., the system bifurcates from one steady state to other steady states. The points of the state-parameter space R^{n+m} that represent equilibria of Eq. (1) are obtained via the equation

$$\boldsymbol{Y}(\boldsymbol{y}, \mu) = 0 \qquad (2)$$

which defines an m-dimensional manifold representing, in general, a complicated equilibrium surface in R^{n+m}. The equilibrium points on this surface may be stable or unstable. The stable equilibria $\boldsymbol{y}^E$ of an initially stable equilibrium path on this surface are associated with eigenvalues of the Jacobian matrix

$$\boldsymbol{Y}_y(\boldsymbol{y}^E, \mu) = \frac{\partial \boldsymbol{Y}(\boldsymbol{y}^E, \mu)}{\partial \boldsymbol{y}} \qquad (3)$$

Received Dec. 11, 1995; revision received July 5, 1996; accepted for publication Oct. 15, 1996; also published in *AIAA Journal on Disc*, Volume 2, Number 2.

*Professor, Department of Structural Analysis and Steel Bridges, 42, Pattission Street. Associate Fellow AIAA.

which are complex conjugate with negative real parts. When the real parts of all eigenvalues are different from zero the equilibrium point is called hyperbolic (generic case). Clearly, the origin $\boldsymbol{y}^E = 0$ is an equilibrium point regardless of the value of μ, i.e., $\boldsymbol{Y}(0, \mu) = 0$.

Considering steady-state solutions of Eq. (1), associated with equilibria and periodic motions, a bifurcation point occurs when 1) a hyperbolic equilibrium point of Eq. (2) becomes nonhyperbolic (at least one eigenvalue of the Jacobian at that point has zero real parts), 2) a hyperbolic periodic orbit[9] becomes nonhyperbolic (at least one characteristic multiplier has unit modulus), and 3) there exist trajectories connecting saddle points.

Bifurcations of the first and sometimes of the second case are local; namely, they can be explored via a local (linear) analysis based on the Jacobian matrix (3). However, the bifurcations of the third case are global, which, in general, can be explored by employing a global (nonlinear) analysis.[4]

III. Local Analysis

It can be shown that the characteristic equation of the Jacobian matrix (3) evaluated at an equilibrium state is given by[10]

$$\left| \rho^2[\alpha_{ij}] + \rho[c_{ij}] + [V_{ij}] \right| = 0 \tag{4}$$

where $[\alpha_{ij}]$ is the positive definite matrix associated with the total kinetic energy of the system, $[c_{ij}]$ is a nonnegative definite dissipative matrix, and $[V_{ij}]$ is, in general, an asymmetric matrix for nonpotential systems. Setting its determinant equal to zero, i.e.,

$$\det[V_{ij}] = |V_{ij}| = 0 \tag{5}$$

we establish the static buckling (divergence) equation from which we obtain the set of buckling loads.

As assumed earlier, geometrically perfect, system bifurcations (static or dynamic) are associated with $\boldsymbol{y}^E = 0$. Hence, the static buckling equation (5) is equivalent to

$$\det \boldsymbol{Y}_y(0, \mu) = 0 \tag{6}$$

Equation (4) after expansion yields

$$f(\rho) = \rho^n + a_1\rho^{n-1} + a_2\rho^{n-2} + \dots a_{n-1}\rho + a_n = 0 \tag{7}$$

where

$$a_1 = -\operatorname{tr} \boldsymbol{Y}_y = \sum_{i=1}^{k} \tilde{c}_{ii} = \sum_{i=1}^{n} \rho_i$$
$$a_n = \det \boldsymbol{Y}_y = \det[\tilde{V}_{ij}] = \prod_{i=1}^{n} \rho_i \tag{8}$$

with $[\tilde{c}_{ij}] = [\alpha_{ij}]^{-1}[c_{ij}]$, $[\tilde{V}_{ij}] = [\alpha_{ij}]^{-1}[V_{ij}]$, and ρ_i $(i = 1, \dots, n)$ the Jacobian eigenvalues. The coefficients a_i $(i = 1, \dots, n)$ can be determined by means of Bôcher's formula.[12] It can be seen that $a_1 = a_1(\tilde{c}_{ij})$ being always positive, $a_2, \dots, a_{n-1}$ are functions of λ, η, k_i, and $\tilde{c}_{ij}$, whereas $a_n = a_n(\lambda, \eta, k_i)$. It is clear that the buckling equation (5) or (6) (implying $a_n = 0$) yields an $n/2$-order algebraic polynomial with respect to λ^c (critical load). The roots of this polynomial depend on η and k_i.

The local bifurcations can be studied by discussing the nature of eigenvalues of the linearized system associated with the Jacobian matrix (3). The nature of the Jacobian eigenvalues depends on the control parameter vector μ. By varying μ smoothly, some eigenvalues may be subjected to sudden changes, producing a variety of bifurcational phenomena. For small values of the loading λ (keeping constant the other parameters), the initial path is stable and the corresponding hyperbolic equilibria have complex conjugate Jacobian eigenvalues with negative real parts. As λ increases (while the other parameters remain constant) at a bifurcation point the corresponding critical equilibrium state becomes nonhyperbolic. Three characteristic cases may basically occur: 1) a nonhyperbolic equilibrium point with one zero (real) eigenvalue, 2) a nonhyperbolic equilibrium point with a pair of purely imaginary eigenvalues, and 3) a nonhyperbolic equilibrium point with a double zero (real) eigenvalue (associated with a defective Jacobian). In cases 1 and 3 the Jacobian becomes singular.

The first case is associated with the so-called divergence instability and can be also studied via a nonlinear static analysis. The second case is a typical case of dynamic instability (called also flutter or self-excited vibrations); then the system bifurcates into limit cycles, stable (periodic attractor) or unstable. Note that flutter according to the classical (linear) analysis is a divergent motion; however, using a nonlinear analysis it can be proved to be of bounded amplitude (stable limit cycles). The third case (double zero eigenvalue) is associated with limit cycles, although the corresponding dynamic bifurcation occurs at a static buckling (divergence) load (see Arnold–Bogdanov bifurcation[11]). Bifurcations associated with a point attractor are static, whereas those associated with a periodic attractor (limit cycles) are dynamic.

The buckling loads λ^c for which $a_n = 0$ also render the Jacobian singular with one zero eigenvalue [see Eq. (7)]. Note that the Jacobian may be associated with a pair of purely imaginary eigenvalues if such a pair satisfies Eq. (7). When this happens in a region of existence of adjacent equilibria for a certain value of λ, say $\lambda = \lambda_{\text{cr}}$, such that $\lambda_{\text{cr}} < \lambda^c_{(1)}$, then dynamic instability occurs prior to divergence.[7] This is a Hopf (local) dynamic bifurcation (normally appearing in regions of nonexistence of adjacent equilibria), which is established via the equation[12]

$$\Delta_{n-1} = (-1)^{n(n-1)/2} \prod_{i<j}^{1,\dots,n} (\rho_i + \rho_j) = 0 \tag{9}$$

where Δ_{n-1} is the Routh–Hurwitz determinant of order $n-1$ associated with Eq. (7). It was shown that the last equation may be satisfied for certain relations between the damping coefficients that may be of negligibly small magnitude (i.e., practically for an undamped system). The Jacobian becomes singular in case of a double zero eigenvalue occurring when $a_{n-1} = a_n = 0$, being also possible for certain relations of the damping coefficients[7] rendering $a_{n-1} = 0$.

In the first case ($a_n = 0$), we have bifurcation into hyperbolic (stable or unstable) equilibria (static bifurcation), whereas in the second ($\Delta_{n-1} = 0$) and third ($a_{n-1} = a_n = 0$) cases we have bifurcation (stable or unstable) into limit cycles (dynamic bifurcations).

The boundary between the regions of existence and nonexistence of adjacent equilibria, being a double critical point (η_0, λ^c_0), can be established following the procedure outlined by Kounadis,[4,7] namely, by solving the system of algebraic equations

$$a_n(\lambda^c, \eta; k_i) = a_{n\lambda}(\lambda^c, \eta; k_i) = 0 \qquad (a_{n\lambda} = \partial a_n / \partial \lambda) \tag{10}$$

with respect to λ^c and η for given values of the parameters k_i.

Let λ^c_0 be the smallest positive value of λ^c obtained by solving Eqs. (10) for which $\eta(\lambda^c_0; k_i) = \eta_0$. It is clear that η_0 is either a maximum or a minimum of the function (curve) $\eta = \eta(\lambda^c; k_i)$ provided that $a_{n\lambda\lambda}(\lambda^c, \eta_0; k_i) \neq 0$. The boundary between divergence and flutter instability corresponds to the tangent at the double point (λ^c_0, η_0) of the curve η vs λ. Since η_0 depends on k_i [i.e, $\eta_0 = \eta_0(k_i; \lambda^c_0)$], it is important to find the extreme value of η_0 that renders maximum the region of divergence instability. This can be established very conveniently by employing the existence theorem for implicit functions of several variables.

Indeed, if $a_n(\lambda^c, \eta, k_i)$ is a function continuous together with its partial derivatives in a neighborhood of a point $(\lambda^c_0, \eta_0, k^0_i)$ such that

$$a_n(\lambda^c_0, \eta_0, k^0_i) = 0, \qquad a_{n\eta}(\lambda^c_0, \eta_0, k^0_i) \neq 0 \tag{11}$$

then $a_n(\lambda^c, \eta, k_i) = 0$ possesses a unique solution $\eta = \eta(\lambda^c; k_i)$ defined in a sufficiently small neighborhood of the point (λ^c_0, k^0_i) depending continuously on λ^c and k_i and satisfying $\eta(\lambda^c_0, k^0_i) = \eta_0$. Moreover, the function $\eta = (\lambda^c, k_i)$ has continuous derivatives. Introducing the last function into $a_n = 0$, we obtain the identity

$$a_n\left[\lambda^c, \eta(\lambda^c, k_i), k_i\right] \equiv 0 \tag{12}$$

The necessary condition for an extremum of the function $\eta = \eta(\lambda^c, k_i)$ is

$$\frac{\partial \eta}{\partial \lambda} = \frac{\partial \eta}{\partial k_i} = 0 \qquad (i = 1, \dots, m-2) \tag{13}$$

Differentiation of identity (12) with respect to λ^c and k_i and using Eq. (13) yields

$$\left.\begin{aligned} a_{n\lambda}(\lambda^c, \eta, k_i) = 0 \\ a_{nk_i}(\lambda^c, \eta, k_i) = 0 \end{aligned}\right\} \qquad i = 1, \ldots, m - 2 \tag{14}$$

Solving the system of Eqs. (14) together with equation

$$a_n(\lambda^c, \eta, k_i) = 0 \tag{15}$$

we obtain the double point $(\lambda_0^c, \eta_0, k_i^0)$ corresponding to the smallest positive values of λ_0^c and k_i^0. This point is an extremum of the function $\eta = \eta(\lambda^c, k_i)$ whose nature is established by examining the sign of the second variation $\delta^2\eta(\lambda^c, k_i^0)$.

The analysis can be substantially simplified by reducing the dimensions of higher-order systems into three or four dimensions that actually capture the qualitative behavior of a system. This can be achieved via the Lyapunov–Schmidt technique as well as via the local techniques of the center manifold,[13] of normal forms,[9] and the splitting lemma.[14] In the sequel we consider a two-degree-of-freedom (four-dimensional) system to include interaction modes phenomena.

IV. Four-Dimensional System

This corresponds to a two-degree-of-freedom statical system. As a typical illustrative example, we consider the model of Ziegler shown in Fig. 1 for which many numerical results are available. In contrast with previous studies in the present analysis the spring stiffnesses k_1 and k_2 of the rotational springs are of different magnitude. The viscous damping coefficients of the corresponding dashpots are β_1 and β_2, respectively. Lagrange equations of motion in dimensionless form are given by[15]

$$\begin{aligned} &(1 + m)\ddot{\theta}_1 + \ddot{\theta}_2\cos(\theta_1 - \theta_2) + \dot{\theta}_2^2\sin(\theta_1 - \theta_2) \\ &\quad + (\beta_1 + \beta_2)\dot{\theta}_1 - \beta_2\dot{\theta} + V_1 = 0 \\ &\ddot{\theta}_2 + \ddot{\theta}_1\cos(\theta_1 - \theta_2) - \dot{\theta}_1^2\sin(\theta_1 - \theta_2) + \beta_2\dot{\theta}_2 \\ &\quad - \beta_2\dot{\theta}_1 + V_2 = 0 \end{aligned} \tag{16}$$

where $m = m_1/m_2$ and

$$\begin{aligned} V_1 &= (1 + k)\theta_1 - \theta_2 - \lambda\sin[\theta_1 + (\eta - 1)\theta_2] \\ V_2 &= \theta_2 - \theta_1 - \lambda\sin\eta\theta_2 \\ \lambda &= P\ell/k_2 \qquad \text{and} \qquad k = k_1/k_2 \end{aligned} \tag{17}$$

Apparently $\mu = (\lambda^c, \eta, k)$. The dimensionless time T is equal to $t(k_2/m_2\ell^2)^{1/2}$.

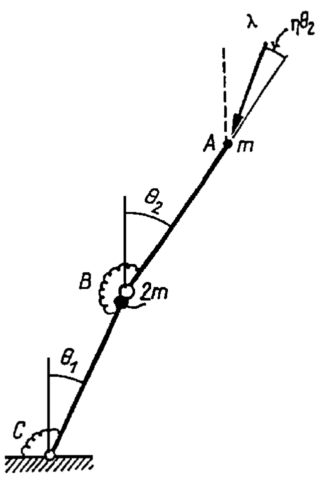

Fig. 1 Ziegler's dissipative model under partial follower load.

The matrices $[\alpha_{ij}]$ and $[V_{ij}]$ evaluated at the trivial state ($\theta_1 = \theta_2 = 0$), as well as the damping matrix $[c_{ij}]$, are given by

$$[\alpha_{ij}] = \begin{bmatrix} m+1 & 1 \\ 1 & 1 \end{bmatrix}, \qquad [V_{ij}] = \begin{bmatrix} k+1-\lambda & -1-\lambda(\eta-1) \\ -1 & 1-\eta\lambda \end{bmatrix}$$

$$[c_{ij}] = \begin{bmatrix} c_{11} & c_{12} \\ c_{12} & c_{22} \end{bmatrix} = \begin{bmatrix} \beta_1+\beta_2 & -\beta_2 \\ -\beta_2 & \beta_2 \end{bmatrix} \tag{18}$$

The characteristic equation based on Eq. (4) is

$$\rho^4 + a_1\rho^3 + a_2\rho^2 + a_3\rho + a_4 = 0 \tag{19}$$

where

$$\begin{aligned} a_1 &= (1/m)[\beta_1 + (m + 4)\beta_2] \\ a_2 &= (1/m)[\beta_1\beta_2 + k + m + 4 - \lambda(2 + m\eta)] \\ a_3 &= (1/m)[\beta_1(1 - \eta\lambda) + \beta_2(k - 2\eta\lambda)] \\ a_4 &= (1/m)[\eta\lambda^2 - \eta\lambda(k + 2) + k] \end{aligned}$$

Previous classical results have been derived for $m = 2$ and $k = 1$ (Refs. 3 and 7). Because m, β_1 and β_2 are both positive, then a_1 is always positive.

For small values of λ we have assumed that the trivial state is stable. This is assured when the characteristic equation (19) has complex conjugate eigenvalues with negative real parts; namely, when Eq. (19) admits the roots $-p_i \pm q_i j$, where $p_i > 0$, $q_i > 0$ ($i = 1, 2$) and $j = \sqrt{-1}$. According to the Routh–Hurwitz criteria, the last roots occur if $a_i > 0$ for all i (necessary condition) and according to Eq. (9),

$$\Delta_3 = (a_1a_2 - a_3)a_3 - a_1^2a_4 > 0 \qquad \text{(sufficient condition)} \tag{20}$$

Note that the effect of m on λ_{cr} was discussed in a recent paper.[7] Obviously, the signs of a_2, a_3, and a_4, as well as the sign of the last inequality, depend on λ for fixed values of η, k, β_1, β_2, and m. As the loading increases slowly, one or more of the coefficients a_2, a_3, and a_4 or inequality (20) may become equal to zero. This violates the satisfaction of the Routh–Hurwitz stability criteria. Indeed, as will be shown, one—at least—root (eigenvalue) does not have a negative real part. It is interesting to discuss whether such a situation is always associated with a bifurcation (static or dynamic).

The dependence of the Jacobian eigenvalues on the sign of the preceding quantities is more conveniently studied by writing Eq. (19) as follows:

$$(\rho^2 + 2B_1\rho + C_1)(\rho^2 + 2B_2\rho + C_2) = 0$$

Clearly,

$$\rho_{1,2} = -B_1 \pm \sqrt{B_1^2 - C_1}, \qquad \rho_{2,3} = -B_2 \pm \sqrt{B_2^2 - C_2} \tag{21}$$

where

$$\begin{aligned} 2B_1 + 2B_2 = a_1 > 0 \qquad & C_1 + C_2 + 4B_1B_2 = a_2 \\ 2B_1C_2 + 2B_2C_1 = a_3 \qquad & C_1C_2 = a_4 \end{aligned} \tag{22}$$

The boundary between divergence (static) and flutter (dynamic) instability, being independent of the mass ratio and of the damping coefficients β_1 and β_2, is determined via the system of equations

$$a_4 = a_{4\lambda} = 0 \tag{23}$$

from which we get the double critical point

$$\eta_0 = \frac{4k}{(k+2)^2}, \qquad \lambda_0^c = \frac{k+2}{2} \tag{24}$$

For $k = 1$ we find $\eta_0 = \frac{4}{9}$ and $\lambda_0^c = \frac{3}{2}$ (Refs. 3, 7, and 15). From Eqs. (24) it is deduced that the maximum value of η_0 corresponds to $k = 2$ and is equal to $\frac{1}{2}$ (i.e., $\max\eta_0 = \frac{1}{2}$). It is also found that $\eta_0 \to 0$ either for $k \to 0$ or for $k \to \infty$.

From the preceding observations it is apparent that the double critical point (24) varies from $\eta_0 \to 0$ to $\eta_0 = 0.50$. Because η

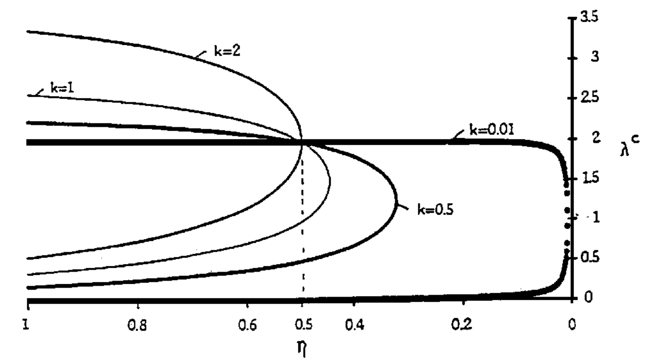

Fig. 2 Four curves η vs λ^c for $k = 0.01, 0.50, 1,$ and 2.

varies normally from $\eta = 0$ (tangential load) to 1 (conservative load), it is clear that for $k \to 0$ or ∞ the region of divergence instability becomes maximum (because $\eta_0 \to 0$), whereas at the same time the region of nonexistence of adjacent equilibria tends to disappear. On the other hand, for $k = 2$ the last region becomes maximum (i.e., $0 < \eta < 0.50$), whereas the region of divergence for $\eta < 0.50$ disappears (then $0.50 < \eta < 1$). For $k > 2$ the region of divergence for $\eta < 0.50$ reappears and increases with increasing k. From Fig. 2 we can see four curves η vs λ^c for $k = 0.01, 0.5, 1,$ and 2. Note that all these curves pass through the extreme double point ($\eta_0 = \frac{1}{2}$ and $\lambda^c = 2$) that corresponds to the fully fixed cantilever.[6]

Using $a_4 = 0$ one could also consider η as a function of two parameters; i.e., $\eta = \eta(\lambda^c, k)$. Then, application of Eqs. (13) yields $\lambda_0^c = 2$ and $k^0 = 2$, which imply $\eta_0 = 0.50$. Obviously, application of Eqs. (11) is possible because $\alpha_{n\eta}(\lambda_0^c, \eta_0, k_0) = -2 \neq 0$. However, the point ($\lambda_0^c = 2, k^0 = 2$) is not an extremum of the surface $\eta = \eta(\lambda^c, k)$; it is a saddle point. This is so because the second variation $\delta^2\eta$ is indefinite; indeed, one can readily show that the matrix

$$\begin{bmatrix} \dfrac{\partial^2\eta}{\partial\lambda^2} & \dfrac{\partial^2\eta}{\partial\lambda\partial k} \\ \dfrac{\partial^2\eta}{\partial k\partial\lambda} & \dfrac{\partial^2\eta}{\partial k^2} \end{bmatrix} = \begin{bmatrix} \dfrac{1}{4} & -\dfrac{1}{8} \\ -\dfrac{1}{8} & 0 \end{bmatrix} \tag{25}$$

has a negative determinant.

Discussion of Eigenvalues

In case of an initially stable path associated with eigenvalues, $-p_i \pm jq_i$ (where $p_i > 0$, $q_i > 0$, and $j = \sqrt{-1}$), one can show that

$$p_i = B_i \quad \text{and} \quad C_i = \sqrt{p_i^2 + q_i^2} \qquad (i = 1, 2) \tag{26}$$

Subsequently, we are seeking cases where dynamic instability may occur prior to divergence; namely, for $\lambda = \lambda_{cr} < \lambda_{(1)}^c$, where $\lambda_{(1)}^c$ is the smallest buckling (divergence) load obtained via $a_4 = 0$. To this end we examine the sign of Δ_3, a_2, a_3, and a_4 as λ increases slowly from zero assuming that all these quantities are positive for small values of λ (initially stable equilibrium path).

Case 1: $\Delta_3 = 0$

If λ_{cr} is the smallest root obtained from $\Delta_3 = 0$, which is smaller than $\lambda_{(1)}^c$, then dynamic buckling (associated with dynamic bifurcation) occurs before static (divergence) buckling. This occurs when either B_1 or B_2 vanishes, and thus the characteristic equation (19) admits a pair of purely imaginary eigenvalues $\pm jq_i$ for $i = 1$ or 2 (transformation of a hyperbolic to a nonhyperbolic equilibrium point). Let B_1 be zero. Then from Eqs. (22) we get

$$\begin{aligned} 2B_2 &= a_1 > 0 \qquad & C_1 + C_2 &= a_2 \\ 2B_2C_1 &= a_3 \qquad & C_1C_2 &= a_4 \end{aligned} \tag{27}$$

Elimination of B_2, C_1, and C_2 from Eqs. (27) yields $\Delta_3 = 0$, which due to relations (19) gives

$$A\lambda^2 + B\lambda + \Gamma = 0 \qquad (\text{with } B^2 \geq 4A\Gamma) \tag{28}$$

or

$$\lambda^2 + (B/A)\lambda + (\Gamma/A) = 0 \qquad (A \neq 0) \tag{28'}$$

where

$$\begin{aligned} A &= \eta[(\beta+2)^2 - (m+2)^2 + m\eta(\beta+2)(m+2)] \\ B &= -2(\beta+m+4)(\beta+k) - \eta\{m(\beta+k)(m-\beta) \\ &\quad + (m+\beta+4)[\beta_1\beta_2(\beta+2) + (\beta-k)(m+2)]\} \\ \Gamma &= 4\beta^2 + \beta m^2 + \beta k^2 + 4k^2 + 16\beta + 8\beta m \\ &\quad - 2\beta km + (\beta+m+4)(\beta+k)\beta_1\beta_2 \end{aligned} \tag{29}$$

The equation $\Delta_3 = 0$ or Eq. (28′) yields Hopf bifurcations. For $m = 2$, $\beta = 1$, and $k = 1$, we find[7]

$$A = \eta(24\eta - 7), \qquad B = -28 - \eta(4 + 21\beta_1\beta_2)$$
$$\Gamma = 41 + 14\beta_1\beta_2$$

We are looking for positive roots of Eq. (28′) $\lambda = \lambda_{cr} > 0$. Equation (28′) has both roots positive if $B/A < 0$ and $\Gamma/A > 0$, whereas if $\Gamma/A < 0$ we have only one positive root being equal to

$$\lambda_{cr}^{(1)} = \tfrac{1}{2}\left[-(B/A) + \sqrt{(B^2/A^2) - (4\Gamma/A)}\right] > 0 \tag{30}$$

If $B/A < 0$ and $B^2 = 4\Gamma$, then

$$\lambda_{cr} = -(B/2A) > 0 \tag{31}$$

Via Eq. (28′) one can search positive values of $\lambda_{cr} = \lambda_{cr}(\beta, k, m, \eta)$ that are smaller than $\lambda_{(1)}^c$ (first static buckling load obtained via $a_4 = 0$) or for the extreme case (at $\eta = \eta_0$) we are looking for $\lambda_{cr} < \lambda_0^c$ (where due to $a_4 = 0$, $\lambda_{(1)}^c < \lambda_0^c$). Then λ_{cr} must be smaller than $(k+2)/2$ [see Eq. (24)]. If such positive values λ_{cr} exist, then dynamic instability occurs prior to divergence. Clearly, for $\lambda < \lambda_{cr}$ the trivial state is asymptotically stable (associated with a point attractor), whereas for $\lambda > \lambda_{cr}$ the primary path ceases to be stable because the real part of one of the two pairs of complex conjugate eigenvalues becomes positive.

At a Hopf bifurcation the response of a system associated with a hyperbolic equilibrium point bifurcates into stable or unstable limit cycles.

Case 2: a_2 or $a_3 = 0$

If either a_2 or a_3 vanishes for a certain $\lambda = \lambda_{cr}^* < \lambda_{(1)}^c$, then due to relations (22) and (26), B_1 and B_2 are of opposite sign. Then, dynamic instability proceeds to divergence instability. In case $a_2 = 0$, we find

$$\lambda_{cr} = \frac{\beta_1\beta_2 + k + m + 4}{2 + m\eta} \tag{32}$$

which is less than $\lambda_{(1)}^c$ or in the extreme case less than λ_0^c (where $\lambda_{(1)}^c < \lambda_0^c$). Then, due to Eq. (24),

$$\frac{\beta_1\beta_2 + k + m + 4}{2 + m[4k/(k+2)^2]} < \frac{k+2}{2}$$

The last inequality is satisfied when

$$k > 2 \qquad \text{and} \qquad m > [(k+2)/(k-2)](2 + \beta_1\beta_2)$$

On the other hand $a_3 = 0$ is satisfied when, due to Eqs. (19),

$$\beta(1 - \eta\lambda) + k - 2\eta\lambda = 0 \qquad (\beta = \beta_1/\beta_2)$$

or

$$\lambda_{cr} = \frac{\beta + k}{\eta(\beta + 2)} \tag{33}$$

Then, dynamic instability occurs prior to divergence if $\lambda^*_{cr} < \lambda^c_0$ or

$$\frac{\beta + k}{[4k/(k+2)^2](\beta+2)} < \frac{k+2}{2}$$

or

$$k^2 - (\beta + 2)k + 2\beta < 0 \tag{34}$$

The last inequality is satisfied for values of k between the roots β and 2. If $\beta > 2$, then $2 < k < \beta$, whereas if $\beta < 2$, then $\beta < k < 2$.

We now discuss whether at the critical state of case 2 a hyperbolic equilibrium point being stable for $\lambda < \lambda^*_{cr}$ becomes unstable for $\lambda \geq \lambda^*_{cr}$. This is not the case since for a_2 or a_3 tending to zero (0^+), $\Delta_3 < 0$, which means that for λ slightly smaller than λ^*_{cr} the corresponding trivial state is unstable. Then it is also deduced that for a certain λ smaller than λ^*_{cr}, $\Delta_3 = 0$ (Hopf bifurcation). The critical load λ_{cr} associated with $\Delta_3 = 0$ is smaller than the value of λ $(=\lambda^*_{cr})$, which renders a_2 or a_3 equal to zero. However, by step increasing λ above λ_{cr} (associated with $\Delta_3 = 0$), it is worth discussing numerically whether a dynamic bifurcation corresponds to $a_2 = 0$ or $a_3 = 0$. Thus, in case 2 the precritical, critical, and postcritical states ($\lambda < \lambda^*_{cr}$ or $\lambda \geq \lambda^*_{cr}$) are associated with hyperbolic equilibrium points (having one eigenvalue with positive real part). The corresponding trivial states are locally unstable, but we do not know whether the system is globally stable.

Case 3: $a_4 = 0$

This is the typical critical state associated with divergence instability. A hyperbolic equilibrium point becomes nonhyperbolic at the critical state; then a pair of complex conjugate eigenvalues with negative real part is transformed to a zero eigenvalue and to a negative eigenvalue. Clearly, when $a_4 = 0$, due to Eqs. (22) C_1 or C_2 becomes zero, which in view of Eqs. (21) implies either $\rho_1 = 0$ and $\rho_2 = -B_1$ or $\rho_3 = 0$ and $\rho_4 = -B_2$. The static critical loads are obtained via the equation $a_4 = 0$, yielding

$$\lambda^c_{1,2} = \tfrac{1}{2}\left[k + 2 \pm \sqrt{(k+2)^2 - (4k/\eta)}\right] \tag{35}$$

The smallest (critical) of these loads $\lambda^c_{(1)}$ obtained also via a static analysis (or using Ziegler's criterion) is meaningful for structural design purposes when dynamic instability does not occur for a load $\lambda = \lambda_{cr}$ smaller than $\lambda^c_{(1)}$.

Note that for a load λ slightly greater than $\lambda^c_{(1)}$, a_4 becomes negative (excluding the case where $\eta = \eta_0$ and $\lambda = \lambda^c_0$), implying that either C_1 or C_2 becomes negative, which due to Eqs. (21) yields one positive root. Namely, the trivial state is (locally) unstable, but the system may be globally stable or unstable depending on the static stability or instability of the critical (divergence) state. In the first case (stability), the system exhibits a point attractor, whereas in the second one (instability) is subject to dynamic buckling associated with an escaped motion via the saddle point of the trivial state.[10]

Finally, note that the double critical point (η_0, λ^c_0) does not have the preceding salient features of an equilibrium point of divergence instability. This is so because at the double equilibrium point $\eta_0 = 4k/(k+2)^2$ we have

$$a_4 = (\eta_0/m)[\lambda^2 - \lambda(k+2) + (k/\eta_0)]$$

or

$$a_4 = \frac{4k}{m(k+2)^2}\left[\lambda - \left(\frac{k+2}{2}\right)\right]^2 \tag{36}$$

Clearly, regardless of whether $\lambda > \lambda^c_0 = (k+2)/2$ or $\lambda < \lambda^c_0$, the quantity a_4 does not change sign. Excluding the case $k\beta = 1$ (implying $a_3 = 0$), for $\eta = \eta_0$ and for λ sufficiently, smaller than λ^c_0, Eq. (19) has two pairs of complex conjugate roots (initially stable path), and hence from Eq. (21) it follows that $B_i^2 < C_i$ $(i = 1, 2)$. Obviously, $a_4 > 0$ for all $\lambda \neq \lambda^c_0$ and $a_4 = 0$ only for $\lambda = \lambda^c_0$, which in turn implies that one C_i $(i = 1, 2)$, say C_1, becomes zero. Then from Eq. (21) it follows that Eq. (19) has one zero root, one negative root, and a pair of complex conjugate roots with negative real part. One now observes that for a certain value of λ slightly smaller than λ^c_0, $C_1 = C_1(\lambda^c)$ becoming very small violates the preceding inequality since $B_1^2 > C_1$. Then Eq. (19) has two negative roots

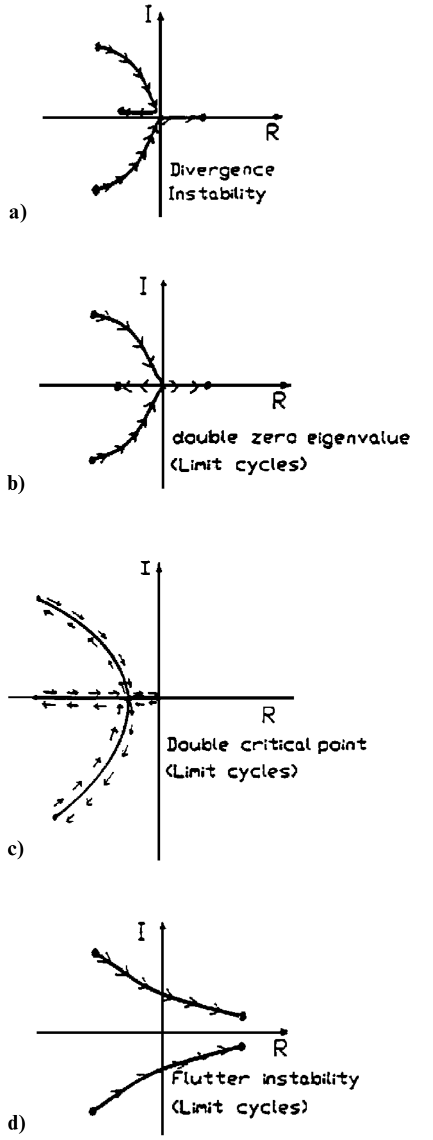

Fig. 3 Characteristic types of instabilities as the eigenvalues vary in the complex plane (*I*, *R*).

(because $a_4 > 0$ for $\lambda \neq \lambda^c_0$) and a pair of complex-conjugate roots with negative real part. Hence, the double point (η_0, λ^c_0) is a hybrid or pseudoequilibrium point associated with (stable) limit cycles[7] (see Fig. 3c).

Case 4: $a_3 = a_4 = 0$

In this case the characteristic equation (19) has a double zero eigenvalue, $\rho_1 = \rho_2 = 0$, and two other roots ρ_3 and ρ_4 given by

$$\rho_{3,4} = \tfrac{1}{2}\left(-a_1 \pm \sqrt{a_1^2 - 4a_2}\right) \tag{37}$$

Because a_2 is usually positive and $a_1 > 0$ (always), ρ_3 and ρ_4 are either both negative or complex conjugate with negative real part (which occurs very frequently due to the very small magnitude of a_1). If $a_2 < 0$, then one eigenvalue is positive yielding (local) instability of the trivial state. Note that for the unrealistic case of an undamped system $a_3 = 0$; hence this system at $\lambda = \lambda^c_{(1)}$ undergoes closed trajectories (implying stability) about the trivial state, which is a center.

Let us now consider the case $a_2 > 0$. Clearly if $a_3 = a_4 = 0$, the Jacobian matrix is singular as in cases 1 and 3 (but not in case 2 where either $a_2 = 0$ or $a_3 = 0$). Moreover, when $a_3 = a_4 = 0$ and Eq. (37) has complex conjugate roots, the Jacobian is a defective matrix (i.e., being not similar to a diagonal matrix). There are three linearly independent eigenvectors because one eigenvector corresponds to the double zero eigenvalue. There is only one postbuckling equilibrium path passing through two consecutive critical points.[7,15] The response of the system at the critical (divergence) state where $a_3 = a_4 = 0$ is associated with limit cycles. This is so because due to Eq. (20), $\Delta_3 = 0$, implying a limit cycle response. This response holds also for loads $\lambda > \lambda_{cr}$ where λ_{cr} is the smallest root of $\Delta_3 = 0$. For higher loads λ [smaller than $\lambda^c_{(2)}$] the system exhibits a point attractor in case of a stable postbuckling equilibrium path. However, for much higher loads the model exhibits again limit cycles.

The analysis when $a_3 = a_4 = 0$ is facilitated by transforming the Jacobian matrix into Jordan canonical form.[7] For Ziegler's model it was shown that the Jacobian matrix is also a nonderogatory matrix as being associated with one Jordan block. The preceding local analysis shows that the trivial state is locally unstable, whereas the global response may be stable or unstable. When $a_3 = a_4 = 0$, Eqs. (19) yield

$$\lambda^c = \frac{k^2 + 2\beta}{k + \beta}, \qquad \eta = \frac{(\beta + k)^2}{(\beta + 2)(k^2 + 2\beta)} \qquad \left(\beta = \frac{\beta_1}{\beta_2}\right) \tag{38}$$

which for $k = 1$ coincide with the corresponding formulas presented in a recent paper by Kounadis.[7]

Obviously, $\beta \longrightarrow 0$ implies $\lambda^c = k$, whereas $\beta \longrightarrow \infty$ gives $\lambda^c = 2$. In both cases $\eta \longrightarrow 0.5$. Note that λ^c coincides with the $\lambda^c_{(1)}$ given in relation (35) if k takes values between β and 2. More specifically, if $\beta < 2$ and $\beta < k < 2$ or $\beta > 2$ and $2 < k < \beta$, the load λ^c given in relation (38) coincides with, $\lambda^c_{(1)}$ obtained from Eq. (35). If $\beta > 2$ and $k > \beta$ (or $k < 2$), then λ^c coincides with $\lambda^c_{(2)}$; similarly if $\beta < 2$ and $k > 2$ (or $k < \beta$), then β^c coincides with $\lambda^c_{(2)}$.

Note that the critical value $\eta_0 = 0.50$ (for smaller values of which various phenomena may appear) is invariant with respect to the values of all other parameters irrespective of the continuity or discreteness of the cantilever model.

Eigenvalues in the Complex ρ Plane

For fixed η the system response depends on one control parameter; namely on λ. Varying smoothly λ the Jacobian eigenvalues ρ_j describe some paths in the complex ρ plane (I, R). If λ is small (less than its critical value) all eigenvalues are in the negative half-plane. As λ increases according to the preceding development, two types of instability may occur: static (divergence) and dynamic (flutter). The first type occurs when at least one pair of complex eigenvalues with negative real part is transformed at $\lambda = \lambda^c_{(1)}$ into a zero eigenvalue and into a negative eigenvalue (Fig. 3a). For λ slightly greater than $\lambda^c_{(1)}$ the zero eigenvalue becomes positive, whereas the negative eigenvalue continues to be negative (decreasing algebraically). One could also consider that divergence instability may occur when a pair of complex conjugate eigenvalues with negative real part coincides at the origin (case of a double zero eigenvalue) and thereafter proceeds in opposite directions on the real axis (Fig. 3b). However, even for vanishing (but nonzero) damping the preceding case is associated with limit cycles and therefore corresponds to a pseudodivergence (or dynamic) instability type. The double critical (pseudoequilibrium) point is also associated with limit cycles. In this case for $k\beta \neq 1$ (namely $a_3 \neq 0$), a pair of complex conjugate eigenvalues with negative real part crosses the negative R axis and thereafter proceeds in opposite directions on this axis (toward zero and toward negative values of R) in the negative half-plane (Fig. 3c). For $\lambda > \lambda^c_0$ the previous paths are followed but in opposite direction. Hence, the double critical point is also associated with a pseudodivergence (or dynamic) instability type. To the knowledge of the author, this is a

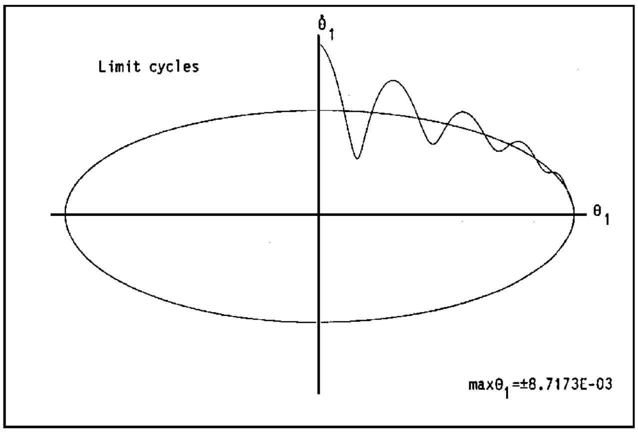

Fig. 4a Phase-plane portrait ($\dot{\theta}_1$ vs θ_1) showing a limit cycle response for $k = 0.50$, $\eta = 0.33$, $\beta = \beta_1/\beta_2 = 0.20$ ($\beta_1 = 0.02$, $\beta_2 = 0.1$), and $\lambda = 0.9222901$.

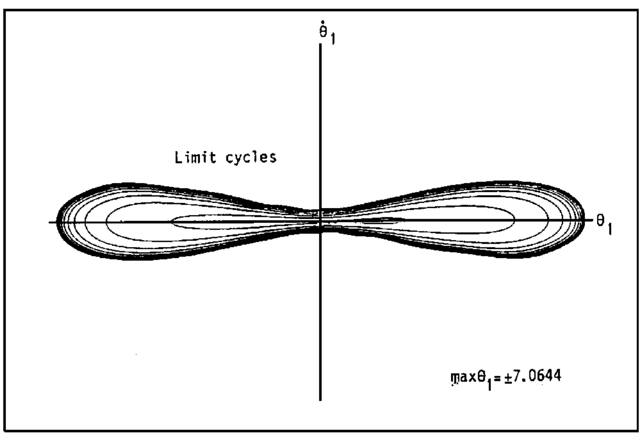

Fig. 4b Phase-plane portrait ($\dot{\theta}_1$ vs θ_1) showing a limit cycle response for $k = 0.50$, $\eta = 0.33$, $\beta = \beta_1/\beta_2 = 0.20$ ($\beta_1 = 0.02$, $\beta_2 = 0.1$), and $\lambda = 1.4675971$.

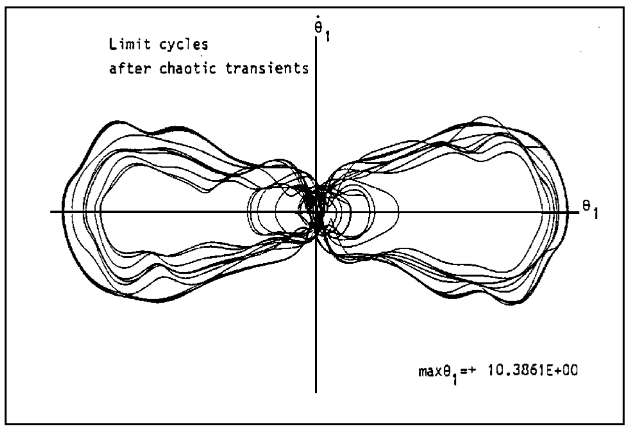

Fig. 4c Phase-plane portrait ($\dot{\theta}_1$ vs θ_1) showing a limit cycle response with chaotic transients for $k = 0.50$, $\eta = 0.33$, $\beta = \beta_1/\beta_2 = 0.20$ ($\beta_1 = 0.02$, $\beta_2 = 0.1$), and $\lambda = 2.4443609$.

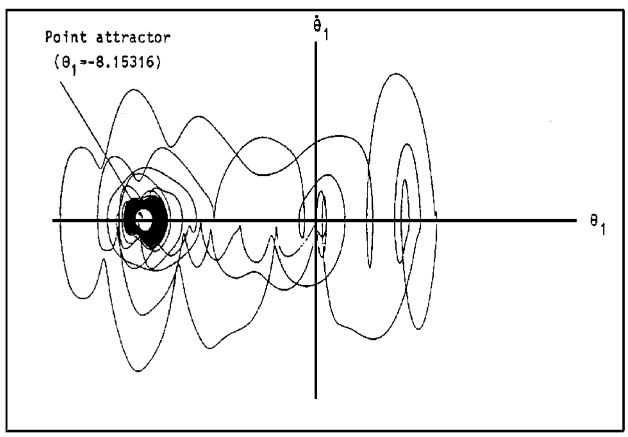

Fig. 4d Phase-plane portrait ($\dot{\theta}_1$ vs θ_1) showing a point attractor response after the decay of chaotic transients for $k = 0.50$, $\eta = 0.33$, $\beta = \beta_1/\beta_2 = 0.20$ ($\beta_1 = 0.02$, $\beta_2 = 0.1$), and $\lambda = 4.00$.

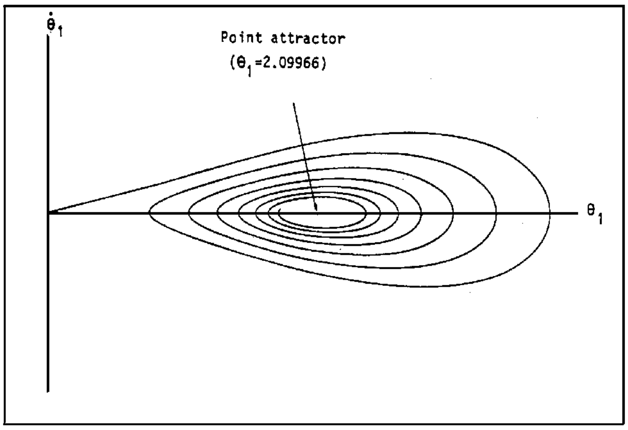

Fig. 5a Phase-plane portrait ($\dot{\theta}_1$ vs θ_1) showing a point attractor response for $k = 0.50$, $\eta = 0.33$, $\beta = \beta_1/\beta_2 = 0.50$ ($\beta_1 = 0.05$, $\beta_2 = 0.1$), and $\lambda = 1.3500$.

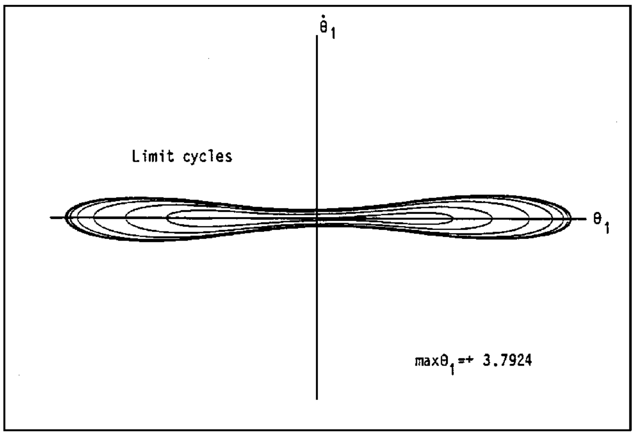

Fig. 5b Phase-plane portrait ($\dot{\theta}_1$ vs θ_1) showing a limit cycle response for $k = 0.50$, $\eta = 0.33$, $\beta = \beta_1/\beta_2 = 0.50$ ($\beta_1 = 0.05$, $\beta_2 = 0.1$), and $\lambda = 1.4675971$.

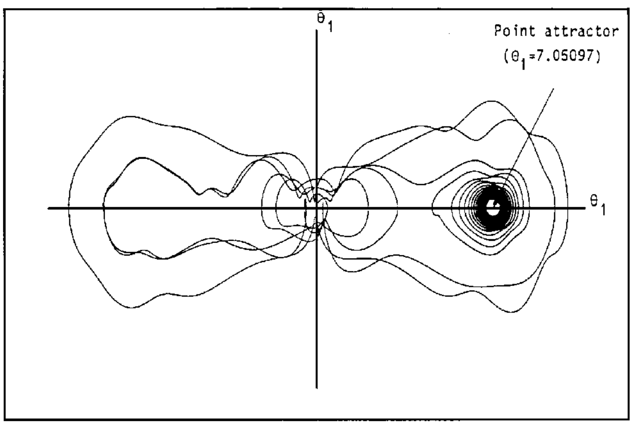

Fig. 5c Phase-plane portrait ($\dot{\theta}_1$ vs θ_1) showing a point attractor response after the decay of chaotic transients for $k = 0.50$, $\eta = 0.33$, $\beta = \beta_1/\beta_2 = 0.50$ ($\beta_1 = 0.05$, $\beta_2 = 0.1$), and $\lambda = 2.4454887$.

new type of dynamic bifurcation. For $k\beta = 1$ and $\lambda = \lambda_0^c$ we have again a double zero eigenvalue.

Finally, dynamic instability associated with a Hopf bifurcation (flutter) occurs when a hyperbolic equilibrium point is transformed into limit cycles (Fig. 3d). In this case a pair of complex conjugate eigenvalues crosses the imaginary axis leading to a pair of complex conjugate eigenvalues with positive real part.

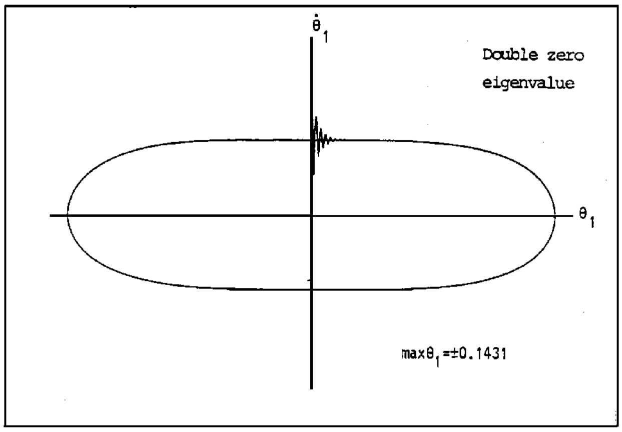

Fig. 6a Phase-plane portrait ($\dot{\theta}_1$ vs θ_1) showing a limit cycle response for $k = 0.50$, $\eta = 0.33$, $\beta = \beta_1/\beta_2 = 0.275$ ($\beta_1 = 0.0275$, $\beta_2 = 0.10$), and $\lambda = \lambda_1^c = 1.03240209$.

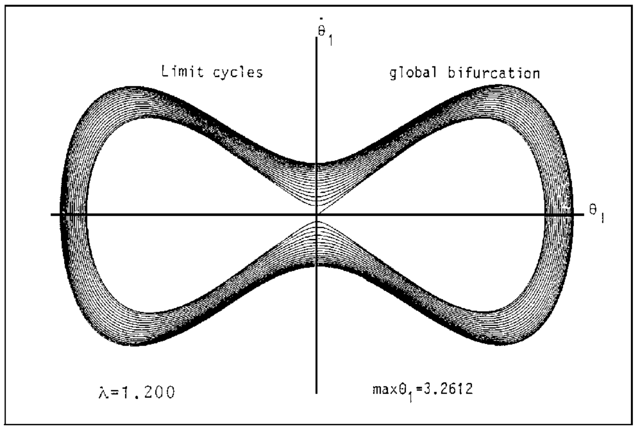

Fig. 6b Phase-plane portrait ($\dot{\theta}_1$ vs θ_1) showing a limit cycle response associated with a global dynamic bifurcation for $k = 0.50$, $\eta = 0.33$, $\beta = \beta_1/\beta_2 = 0.0275$ ($\beta_1 = 0.0275$, $\beta_2 = 0.10$), and $\lambda < \lambda_1^c = 1.200 < \lambda_2^c$.

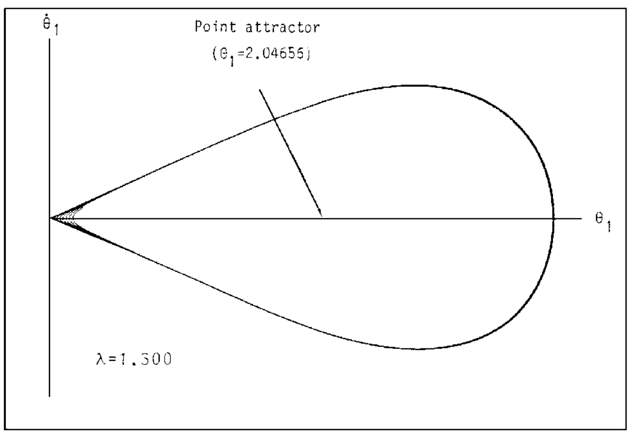

Fig. 6c Phase-plane portrait ($\dot{\theta}_1$ vs θ_1) showing a point attractor response for $k = 0.50$, $\eta = 0.33$, $\beta = \beta_1/\beta_2 = 0.0275$ ($\beta_1 = 0.0275$, $\beta_2 = 0.10$), and $\lambda < \lambda_1^c = 1.200 < \lambda_2^c$.

V. Numerical Results and Discussion

The findings presented recently by Kounadis[7] that were valid for $k = 1$ are qualitatively similar to those of this analysis for $k < 1$ and $k > 1$. The small region of divergence instability ($\frac{4}{9} < \eta < 0.5$) near the double critical point valid for $k = 1$ can increase considerably for $k \to 0$ (or $k \to \infty$) becoming maximum with lower bound $\eta_0 \to 0$ and upper bound $\eta_0 = 0.5$ (which is independent of k). Then, the region of nonexistence of adjacent equilibria disappears completely.

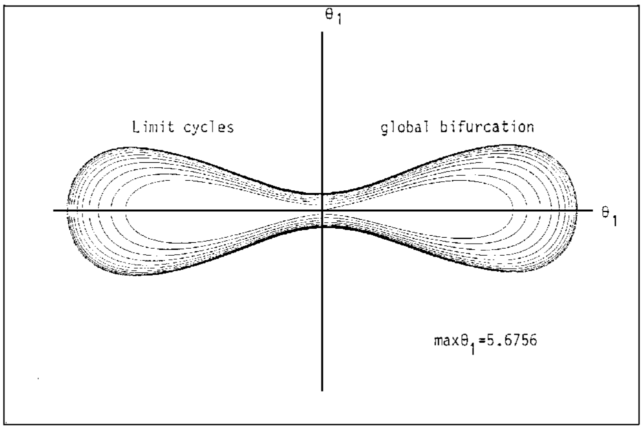

Fig. 6d Phase-plane portrait ($\dot{\theta}_1$ vs θ_1), showing a limit cycle response associated with a global dynamic bifurcaion for $k = 0.50$, $\eta = 0.33$, $\beta = \beta_1/\beta_2 = 0.0275$ ($\beta_1 = 0.0275$, $\beta_2 = 0.10$), and $\lambda < \lambda_1^c = 1.400 < \lambda_2^c$.

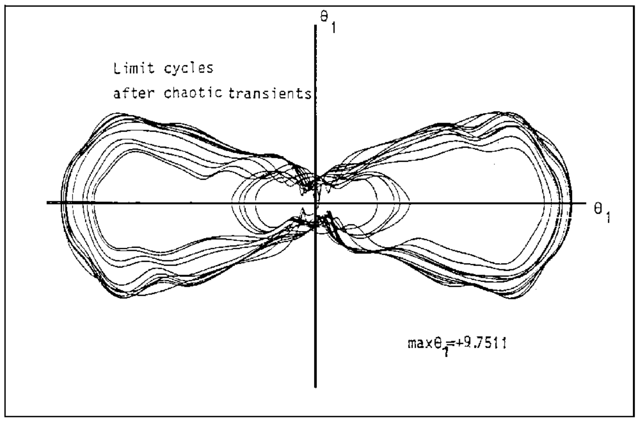

Fig. 6f Phase-plane portrait ($\dot{\theta}_1$ vs θ_1) showing a limit cycle response with chaotic transients for $k = 0.50$, $\eta = 0.33$, $\beta = \beta_1/\beta_2 = 0.0275$ ($\beta_1 = 0.0275$, $\beta_2 = 0.1$), and $\lambda = 2.25$.

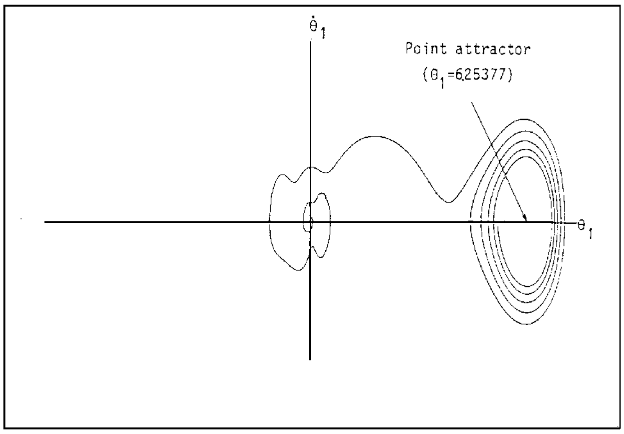

Fig. 6e Phase-plane portrait ($\dot{\theta}_1$ vs θ_1), showing a limit cycle response associated with a global dynamic bifurcation for $k = 0.50$, $\eta = 0.33$, $\beta = \beta_1/\beta_2 = 0.0275$ ($\beta_1 = 0.0275$, $\beta_2 = 0.100$), and $\lambda = 2.00$.

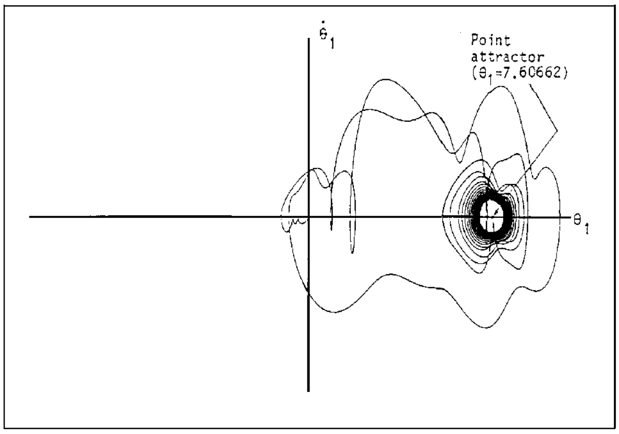

Fig. 6g Phase-plane portrait ($\dot{\theta}_1$ vs θ_1) showing a point attractor response with chaotic transients for $k = 0.50$, $\eta = 0.33$, $\beta = \beta_1/\beta_2 = 0.0275$ ($\beta_1 = 0.0275$, $\beta_2 = 0.10$), and $\lambda = 3.00$.

Since the equation for Hopf bifurcations is of the form

$$\Delta_3(\lambda, \eta;\ k;\ m;\ \beta_1;\ \beta_2) = 0 \tag{39}$$

for $k \longrightarrow 0$ and suitable values of m and $\beta(=\beta_1/\beta_2$ with β_1 and $\beta_2 \longrightarrow 0)$, Eq. (39) can yield $\lambda_{\mathrm{cr}} < \lambda^c_{(1)}$; namely, dynamic instability (via a Hopf bifurcation) may occur in the region $0 < \eta < 0.5$, prior to divergence instability for a nonconservative system in which the region of nonexistence of adjacent equilibria tends practically to zero.

Let us consider the case $k = 0.5$; then Eqs. (24) yield $\eta_0 = 0.32$ and $\lambda^c_0 = 1.25$, and the region of divergence instability is defined by $0.32 < \eta < 0.5$. If we choose $\eta = 0.33 > \eta_0$, $m = 2$, and $\beta = 0.2$, then $\lambda^c_{(1)} = 1.0324$. It can be shown that the smallest root of Eq. (39) is $\lambda_{\mathrm{cr}} = 0.92229$, and hence for $\lambda < \lambda_{\mathrm{cr}}$ the system exhibits a point attractor (associated with the trivial equilibria states). For $\lambda = \lambda_{\mathrm{cr}}$ the system experiences a stable limit cycle response as shown by its phase plane in ($\dot{\Theta}_1$ vs Θ_1) in Fig. 4a. As λ increases, the system continues to exhibit a periodic attractor. The phase plane for $\lambda = \lambda^c_{(2)} = 1.467597$ associated with a periodic attractor is shown in Fig. 4b. The phase plane slightly changes for $\lambda = 2.44436$ for which the coefficient $a_2 = 0$ (Fig. 4c). For a further increase of λ the system experiences again a point attractor (see Fig. 4d for $\lambda = 4$). It is clear that chaotic transients are present for λ much higher than $\lambda^c_{(2)}$.

A different response is observed if $\beta = 0.5$ (instead of $\beta = 0.2$), whereas all the preceding parameters are kept constant. Indeed, for λ sufficiently smaller than $\lambda^c_{(2)}$ the system exhibits a typical point attractor response, regardless of the vanishing of a_3 (occurring at $\lambda = 1.2121212$) or Δ_3 (occurring at $\lambda_{\mathrm{cr}} = 1.250738$). A typical phase-plane portrait is shown in Fig. 5a for $\lambda = 1.35 < \lambda^c_{(2)} = 1.467597$. For λ slightly less than $\lambda^c_{(2)}$ the system experiences a periodic attractor (see Fig. 5b) because this case is associated with one postbuckling path passing through the first and second branching point (see case 4 and Ref. 7). Finally, for higher loads the system exhibits a point attractor due to the existence of a stable postbuckling path (see Fig. 5c).

Case 4 treated earlier (i.e., $a_3 = a_4 = 0$) is obtained if $\beta = 0.2751$, whereas all the other parameters are kept constant. The phase-plane portrait corresponding to a double zero eigenvalue is associated with limit cycles (Fig. 6a). For $\lambda = 1.20$ the system continues to exhibit a limit cycle attractor (Fig. 6b), whereas for $\lambda \geq 1.26$ it experiences a point attractor response (Fig. 6c). For $\lambda = 1.40$ we have again a limit cycle response (Fig. 6d), whereas for $\lambda = 2$ the response of the system is associated with a point attractor (Fig. 6e). For $\lambda = 2.25$ the system exhibits again a limit cycle response (Fig. 6f), whereas for $\lambda \geq 3$ it experiences a point attractor (Fig. 6g). In Fig. 7 representing the curve λ vs Θ_1 and Θ_2 for $k = 0.50$ and $\eta = 0.33$, the alternative regions of periodic and point attractors for various levels of λ occur. Results similar to the preceding results can be obtained for $k \geq 1$. Note that one natural postbuckling path passes through the first and second branching points corresponding to $\lambda^c_{(1)} = 1.0324$ and $\lambda^c_{(2)} = 1.4676$.

From these numerical results, which verify the theoretical findings of this study, one can draw the conclusion that the most important quantities governing the dynamic response of the system are the Routh–Hurwitz determinant Δ_3 and the Jacobian determinant a_4. If either of these quantities is zero for a certain $\lambda*$, the vanishing of any of a_i ($i = 1, 2, 3$) for $\lambda \geq \lambda*$ practically has no effect. Moreover, if Δ_3 vanishes for a λ_{cr} smaller than that for which $a_4 = 0$, then the response of the system for $\lambda > \lambda_{\mathrm{cr}}$ (including the value of λ for which $a_4 = 0$) is associated with limit cycles (see Fig. 8). On

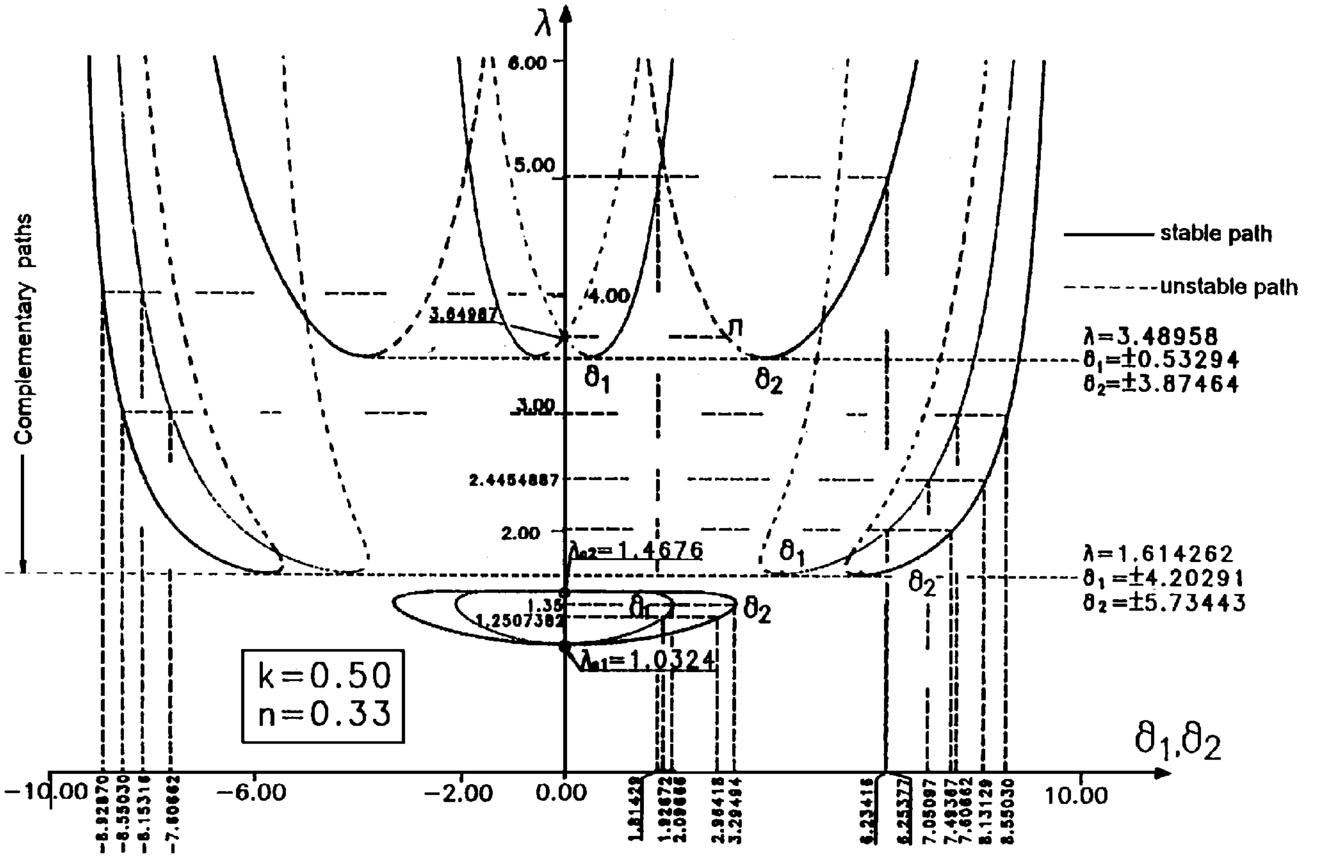

Fig. 7 Natural and complementary equilibrium paths λ vs θ_i (i = 1, 2) and one postbuckling path passing through the first and second branching point. Point and periodic attractors depend on the level of load.

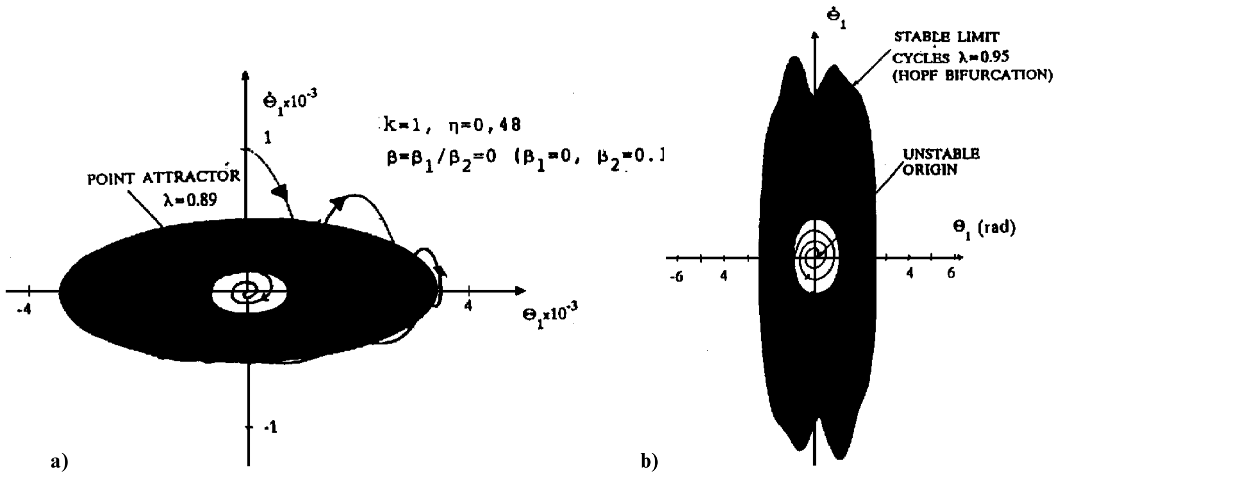

Fig. 8 a) Point attractor for λ = 0.89 and b) stable limit cycles for λ = 0.95 ($<\lambda^c_{(1)}$ = 1.09175171) for k = 1, η = 0.48, β_1 = 0, and β_2 = 0.1.

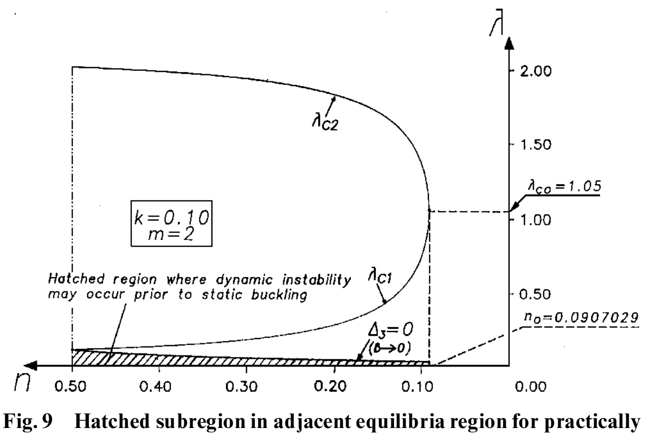

Fig. 9 Hatched subregion in adjacent equilibria region for practically undamped systems, where dynamic instability may occur prior to static buckling.

the other hand, if a_4 vanishes for $\lambda^c_{(1)}$ smaller than that for which $\Delta_3 = 0$, then the system response for λ sufficiently greater than $\lambda^c_{(1)}$ (regardless of the value of λ for which $\Delta_3 = 0$) is associated with a point attractor if the postbuckling path is stable or with a divergent motion (static buckling) if this path is unstable. From Fig. 9 corresponding to a model with $k = 0.1$, $m = 2$, and $\beta_i \longrightarrow 0$ ($i = 1, 2$), one can see a large interval of values of η ($0.50 \lesssim \eta \lesssim 0.20$) for which dynamic buckling may occur prior to static buckling.

Dynamic bifurcations corresponding to loads λ much higher than the critical load that are associated with trajectories passing through or near the origin are global and cannot be established via a linear analysis.

VI. Conclusions

From this study dealing with a two-degree-of-freedom nonlinear dissipative cantilever model under a partial follower load with nonconservativeness loading parameter η ($0 \leq \eta \leq 1$), one can draw the following main conclusions.

1) Irrespective of the continuity or of the discreteness of the model, there is a critical value of η, i.e., $\eta_{cr} = 0.50$, for smaller values of which various phenomena may occur, whereas for $0.50 < \eta \leq 1$ (conservative load) the model response can always be obtained via static stability analyses and Ziegler's kinetic criterion. The value $\eta_{cr} = 0.50$ is invariant with respect to any values of all other parameters.

2) For $\eta \leq 0.50$ the following phenomena may occur.

a) A double critical point (resulting from the coincidence of two consecutive buckling modes) corresponds to a certain value of η, $\eta = \eta_0$, which defines the boundary between the region of existence ($\eta \geq \eta_0$) and the region of nonexistence ($\eta_0 > \eta \geq 0$) of adjacent equilibria. This is the pseudoequilibrium point because it is associated with limit cycles.

b) Even in the region of adjacent equilibria (region of divergence) defined by $\eta_0 < \eta < 0.50$, a practically nondissipative model under certain conditions may lose its stability via flutter (through a Hopf bifurcation) for a load $\lambda = \lambda_{cr}$ smaller than that of divergence instability (i.e., $\lambda_{cr} < \lambda^c_{(1)}$), where λ_{cr} is the smallest load for which the Routh–Hurwitz determinant Δ_3 vanishes. Then the static stability analyses and Ziegler's kinetic criterion (associated with the vanishing of the fundamental circular frequency) fail to predict the actual critical load.

c) If Δ_3 vanishes for a load λ_{cr} greater than $\lambda^c_{(1)}$, the model for $\lambda \geq \lambda^c_{(1)}$ exhibits a point attractor (in case of a stable postbuckling path) up to a certain value $\lambda* < \lambda^c_{(2)}$, whereas for $\lambda \geq \lambda*$ the model exhibits a limit cycle response (due to the existence of one postbuckling path passing through two consecutive branching points). In case of a double zero eigenvalue at $\lambda = \lambda^c_{(1)}$ (implying $\Delta_3 = 0$), the model exhibits for $\lambda \geq \lambda^c_{(1)}$ a limit cycle response, which for greater loads becomes a point attractor response. For much higher loads $\lambda < \lambda^c_{(2)}$, the model experiences again a limit cycle response.

d) Local bifurcations (static or dynamic) are established either via $a_4 = 0$ or $\Delta_3 = 0$.

3) The value of the double critical point $\eta = \eta_0$ depends on the value of a stiffness parameter k. For $k \longrightarrow 0$ or $k \longrightarrow \infty$ it follows $\eta_0 \longrightarrow 0$. In this case the region of existence of adjacent equilibria (region of divergence) defined by $\eta \leq 0.50$ becomes maximum ($0 < \eta \leq 0.5$), whereas the region of nonexistence of adjacent equilibria disappears. On the other hand for $k = 2$ (implying $\eta_0 = 0.5$) the region of nonexistence of adjacent equilibria becomes maximum ($0 \leq \eta \leq 0.50$), whereas the region of divergence defined by $\eta < 0.50$ disappears.

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R. K. Kapania
Associate Editor